

SOME FUNDAMENTAL ASPECTS OF A QUANTUM POTENTIAL

ROBERT CARROLL
UNIVERSITY OF ILLINOIS, URBANA, IL 61801

ABSTRACT. We show that given an essentially arbitrary $Q(x, t, \hbar)$ there are “generalized” quantum theories having Q as their quantum potential.

1. INTRODUCTION

Let a function $Q(x, t, \hbar) \sim Q(x, t) \sim Q$ be given (with properties to be determined). Following [2], in order for Q to be a quantum potential with a Schrödinger equation (SE) (\clubsuit) $-(\hbar^2/2m)\Delta\psi + V\psi = i\hbar\partial_t\psi$ (where $\psi = R(x, t)\exp[iS(x, t)/\hbar]$) one requires that (\spadesuit) $Q = -(\hbar^2/2m)(\Delta R/R)$. We ignore here “delicate” situations where $S = \text{constant}$ etc. (cf. [2, 4, 8]). From (\spadesuit) one derives quantum Hamilton-Jacobi equations (QHJE) of the form

$$(1.1) \quad \partial_t R^2 + \frac{1}{m}\nabla(R^2\nabla S) = 0; \quad \partial_t S + \frac{1}{2m}(\nabla S)^2 + Q + V = 0$$

(V will be assumed to have suitable properties as needed). The plan here is to solve (\spadesuit) for $R = R(Q, f(t), g(t))$ and then fit this into (1.1).

One should note a few known limitations relating quantum and classical mechanics via the quantum potential (cf. [1]). Thus

- (1) for a free particle in 1-dimension (1-D) one has possibilities such as $\psi' = A\exp[i(px - (p^2t/2m))/\hbar]$ and $\psi'' = A\exp[-i(px + (p^2t/2m))/\hbar]$ in which case $Q = 0$ for ψ' and ψ'' separately but for $\psi = (\psi' + \psi'')/\sqrt{2}$ there results $Q = p^2/2m$ ($p \sim \hbar k$ here - cf. Remark 3.1). Hence $Q = 0$ depends on the wave function and cannot be said to represent a classical limit.
- (2) For $V = m\omega^2x^2/2$ and a stationary SE one has solutions of the form $\psi_n(x) = c_n H_n(\xi x)\exp(-\xi^2x^2/2)$ where $\xi = (m\omega\hbar)^{1/2}$, $c_n = (\xi/\sqrt{\pi}2^n n!)^{-1/2}$, and H_n is a Hermite function. One computes that $Q = \hbar\omega[n + (1/2)] - (1/2)m\omega^2x^2$. Hence $\hbar \rightarrow 0$ does not imply $Q \rightarrow 0$ and moreover $Q = 0$ corresponds to $x = \pm\sqrt{(2\hbar/m\omega)[n + (1/2)]}$ so not all systems in quantum mechanics (QM) have a classical limit.

Therefore evidently in general one cannot identify QM as quantization of classical systems or the quantum potential as a vehicle to generate QM since in particular there are physically realizable classical situations that cannot be reached as the

Date: June, 2005.
email: rcarroll@math.uiuc.edu.

limit of some QM system, $\hbar \rightarrow 0$ and $Q = 0$ are generally different concepts, and the condition $Q = 0$ can depend on the wave function. On the other hand we have exhibited and studied in [2, 5, 6] a vast collection of examples and situations where the quantum potential Q in e.g. Schrödinger and Klein-Gordon equations plays a fundamental role in connection with quantum fluctuations, diffusion, Weyl geometry, entropy, etc. In other words there are physical and geometrical origins of quantum potentials and such interaction of QM and geometry is surely related to the elusive understanding of “quantum gravity” (whatever that may be).

2. THE ELLIPTIC EQUATION

Suppose $R = 0$ outside of some region $\Omega \subset \mathbf{R}^3$ (since $R^2 \sim |\psi|^2$ is a probability density this would be reasonable for many QM problems). Consider then the elliptic equation

$$(2.1) \quad L(R) = -\Delta R - \beta QR; \quad \beta = \frac{2m}{\hbar^2}$$

Given say $R \in H_0^1(\Omega)$ this is associated with a bilinear form ($R_i = \partial_i R$)

$$(2.2) \quad B(R, \phi) = \int_{\Omega} \sum R_i \phi_i - \beta \int_{\Omega} QR\phi$$

Recall $\|v\|_{H_0^1}^2 = \|v\|_{L^2}^2 + \sum \|v_i\|_{L^2}^2$ (cf. [3] for notation) so $|B(R, \phi)| \leq c\|R\|_{H_0^1}\|\phi\|_{H_0^1}$ when $Q \in L^\infty$ for example. Further for $\gamma > \beta \sup|Q|$ one has

$$(2.3) \quad B(R, R) + \gamma\|R\|_{L^2}^2 \geq c'\|R\|_{H_0^1}^2$$

so by Lax-Milgram for example one can say that for $\mu \geq \gamma$ there exists a unique solution of $LR + \mu R = 0$ (cf. [3, 7, 9]). On the other hand if e.g. $Q \leq 0$ one has $\beta Q \geq 0$ and

$$(2.4) \quad B(R, R) \geq c''\|R\|_{H_0^1}^2$$

(note for $f \in H_0^1$ one has $\|f\|_{L^2}^2 \leq \hat{c} \sum \|f_i\|_{L^2}^2$). Consequently (cf. [7] for proof)

THEOREM 2.1. For $Q \in L^\infty$ and $Q \leq 0$ the equation $\Delta R = \beta QR$ has a unique solution in H_0^1 . If $Q \in L^\infty$ and $\mu > \beta \sup|Q|$ then there exists a unique solution of $\Delta R + \mu R = \beta QR$ in H_0^1 . Further there is an at most countable set $\Sigma \subset \mathbf{R}$ of eigenvalues $\lambda_k \rightarrow \infty$ such that $\Delta R = \beta QR + \lambda R$ has a unique solution if and only if $\lambda \notin \Sigma$. In particular if $0 \notin \Sigma$ then $-\Delta R = \beta QR$ has a unique solution for any $Q \in L^\infty$.

REMARK 2.1. These are typical results for H_0^1 and using methods of duality and functional analysis one can produce various theorems involving solutions $u \in H^1(\Omega)$ or other Sobolev spaces (cf. [3, 9, 10, 11]). Here one notes that H^{-1} is the dual of H_0^1 where $f \in H^{-1}$ means that there exist $f^i \in L^2$ such that $\langle f, v \rangle = \int_{\Omega} (f^0 v + \sum f^i v_{x_i}) dx$ for $v \in H_0^1$. The theorems for solutions of $\Delta R + \mu R = \beta QR$ or $(L + \mu)R = 0$ above are special cases of $L_\mu u = g$ for say $g \in L^2$ and one can extend easily to see that $L_\mu : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is an isomorphism (cf. [7]). ■

3. THE HAMILTON JACOBI EQUATION IN 1-D

The plan now is to solve for ∇S from the first equation in (1.1) and then reduce the HJ equation to a simple ordinary differential equation in t . This will avoid the need of considering e.g. viscosity solutions of the HJ equation (see e.g. [7]). Thus set first (\blacklozenge) $\nabla S = p$ with $\dot{q} = p/m$ and consider

$$(3.1) \quad m\partial_t R^2 + \nabla(R^2 p) = 0$$

with R given via Theorem 2.1 as $R(x, t) = R(Q(x, t), x)$ (no arbitrary functions of t are introduced here and we recall that Q may depend on \hbar). In 1-D this is $m\partial_t R^2 + \partial(R^2 p) = 0$ from which

$$(3.2) \quad R^2 p = - \int^x m\partial_t R^2 dx + mf(t)$$

(f “arbitrary”). Writing the HJ equation as $\partial_t S + (1/2m)p^2 + Q + V = 0$ one arrives at

$$(3.3) \quad \partial_t S = -Q - V - \frac{1}{2R^2} \left[f(t) - \int^x \partial_t R^2 dx \right]^2$$

from which

$$(3.4) \quad S = - \int^t (Q + V) dt - \frac{1}{2} \int^t \frac{1}{R^4} \left[f(t) - \partial_t \int^x R^2 dx \right]^2 + g(x)$$

for g a suitable “arbitrary” function and $\partial_t F = f$ arbitrary.

THEOREM 3.1. In 1-D, given a solution $R = R(Q(x, t), x)$ of $-\Delta R = \beta QR$ as in Theorem 2.1, one can find a solution $S = S(Q(x, t), x, t)$ in the form (3.4), where f, g are suitable “arbitrary” functions and $V(x)$ is given. This will represent a “generalized” quantum theory in some sense determined by Q (V being a suitable function).

EXAMPLE 3.1. Ignoring temporarily any restriction $R \in H_0^1$ (which is also violated in Items 1 and 2 of Section 1) consider $Q = 0$ so that $R'' = 0$ implies $R = a(t)x + b(t)$ for suitable a, b . Then (3.2) implies (for 1-D and suitable $f(t)$)

$$(3.5) \quad (ax + b)^2 p = -m \int^x \partial_t R^2 + mF_t(t) = -m\partial_t \int^x (ax + b)^2 dx + m\partial_t F(t) \Rightarrow$$

$$\Rightarrow p = \frac{-m}{(ax + b)^2} \partial_t \left[\frac{(ax + b)^3}{2a} \right] + m\partial_t F(t)$$

Now differentiating in x one can write the second equation in (1.1) as ($\dot{p} = \partial_t p$ and $p' = \partial_x p$)

$$(3.6) \quad \dot{p} + \frac{1}{m} pp' + \partial V = 0$$

Consequently one obtains an expression for a putative ∂V in the form

$$(3.7) \quad \partial V = -\dot{p} - \frac{1}{m} pp' = m\partial_t \left\{ \frac{1}{(ax + b)^2} \partial_t \left[\frac{(ax + b)^3}{3a} + F \right] \right\} -$$

$$-m \left\{ \left[\frac{1}{(ax + b)^2} \partial_t \left(\frac{(ax + b)^3}{3a} + F \right) \right] \partial_x \left[\frac{1}{(ax + b)^2} \partial_t \left(\frac{(ax + b)^3}{3a} + F \right) \right] \right\}$$

Consider now special cases a or b equal to zero with $F = 0$. For $a = 0$ one has

$$(3.8) \quad b^2 p = -m \partial_t b^2 x \Rightarrow p = -m x \partial_t \log(b^2) = -2m x \partial_t \log(b)$$

Hence

$$(3.9) \quad \dot{p} = -2m x \partial_t^2 \log(b); \quad p' = -2m \partial_t \log(b);$$

$$\partial V = -\dot{p} - \frac{1}{m} p p' = 2m [\partial_t^2 \log(b) - 2(\partial_t \log(b))^2]$$

Thus if e.g. $b = \exp(\pm ct)$ with $\log(b) = \pm ct$ and $\partial_t \log(b) = \pm c$ one has $(\bullet) \partial V = -4m x c^2$ which seems like a conceivable physical potential $V \sim -2m c^2 x^2$. If $b = 0$ one gets

$$(3.10) \quad a^2 x^2 p = -m \partial_t \int^x a^2 x^2 dx = -\frac{m x^3}{3} \partial_t a^2 \Rightarrow p = -\frac{2m x}{3} \partial_t \log(a)$$

leading to

$$(3.11) \quad \dot{p} = -\frac{2m x}{3} \partial_t^2 \log(a); \quad p' = -\frac{2m}{3} \partial_t \log(a)$$

Hence for $a = \exp(\pm ct)$ as before one has $\partial_t \log(a) = \pm c$ and

$$(3.12) \quad \partial V = -\frac{4m c^2 x}{9} \sim V = -\frac{2}{9} m c^2 x^2$$

much as in the case $a = 0$. One notes that the potential V in both these situations corresponds to the negative of the potential in Item 2 of Section 1. ■

REMARK 3.1. Note that the situation of Item 1 in Section 1 can also be attained here for $Q = 0$. Indeed take $R = 1$ (i.e. $a = 0$ and $b = 1$ in Example 3.1). Then $F \neq 0$ with $R^2 p = p = m \dot{F}(t)$ and since $Q = V = 0$ one has $\partial_t S = -(1/2)F(t)$. Note in Item 1 p is used for $\hbar k$ where k is a frequency and e.g. $S = \hbar k x - (\hbar^2 k^2 / 2m)t$ with $S_t = -\hbar^2 k^2 / 2m \sim -E$ so here $(1/2)F(t) = E$. Then the HJ equation becomes $-E + (1/2m)S_x^2 = 0$ with $S_x = \hbar k$. Alternatively (referring to Item 1) for $\psi = (\psi' + \psi'')/\sqrt{2}$ and $Q = \hbar^2 k^2 / 2m$ with $V = 0$ we have

$$(3.13) \quad R = \sqrt{2} A \cos(kx); \quad R''/R = -k^2; \quad Q = \frac{k^2 \hbar^2}{2m}; \quad S = -\frac{k^2 \hbar^2 t}{2m}; \quad S_t = -\frac{k^2 \hbar^2}{2m}; \quad S_x = 0$$

Consequently $S_t + (1/2m)(S_x)^2 + Q + V = -(k^2 \hbar^2 / 2m) + (k^2 \hbar^2 / 2m) = 0$ and one sees that the same SE can arise from different quantum potentials. ■

REMARK 3.2. Evidently with unique solutions R as in Theorem 2.1 one should arrive at fewer possibilities in the construction of S . Otherwise the map $SE \rightarrow Q$ is seen to be possibly multivalued. ■

REFERENCES

- [1] A. Bolivar, Quantum-classical correspondence, Springer, 2004
- [2] R. Carroll, Fluctuations, information, gravity, and the quantum potential, Springer, to appear, 2005
- [3] R. Carroll, Abstract methods in partial differential equations, Harper-Row, 1969
- [4] R. Carroll, Quantum theory, deformation, and integrability, North-Holland, 2000
- [5] R. Carroll, Fluctuations, gravity, and the quantum potential, gr-qc 0501045 (to appear in a volume on quantum gravity)
- [6] R. Carroll, Information, quantum mechanics, and gravity, Foundations of Physics, 35 (2005), 131-154
- [7] L. Evans, Partial differential equations, American Math. Society, 1998
- [8] A. Faraggi and M. Matone, Inter. Jour. Mod. Phys. A, 14 (2000), 1869-2017
- [9] J. Lions, Équations différentielles opérationnelles, Springer, 1961
- [10] J. Lions, Quelques méthodes de resolution des problèmes aux limites nonlinéaires, Dunod, 1969
- [11] J. Lions and E. Magenes, Nonhomogeneous boundary value problems and applications, Vols. 1-3, Springer, 1972